Mathematics A/3

$$f(x)=\begin{cases} 1 \ \ if \quad 0 \le x < \pi \\ -1 \quad if \ -\pi \le x < 0 \end{cases}, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

$$f(x)=\frac{x}{2} \ \ if \ -\pi \le x < \pi, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

$$f(x)=\begin{cases} \frac{\pi}{2}+x \ if \quad 0 \le x < \pi \\ \frac{\pi}{2}-x \ if \ -\pi \le x < 0 \end{cases}, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

$$f(x)=|x| \ \ if \ -\pi \le x < \pi, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

$$f(x)=\begin{cases} x-1 \ \ if \quad 0 \le x < \pi \\ -1 \quad if \ -\pi \le x < 0 \end{cases}, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

$$f(x)=x^2 \ \ if \ -\pi \le x < \pi, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

$$f(x)=\begin{cases} -\frac{\pi}{2} \quad if \ -\pi \le x < -\frac{\pi}{2} \\ x \quad if \ -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ \frac{\pi}{2} \quad if \ \frac{\pi}{2} \le x < \pi \end{cases}, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

$$f(x)=\begin{cases} -2x+1 \ \ if \quad -\pi \le x < 0 \\ 2x+1 \quad if \quad 0 \le x < \pi \end{cases}, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

$$f(x)=\begin{cases} \frac{x}{2}+1 \ \ if \quad -\pi \le x < 0 \\ 1 \quad if \quad 0 \le x < \pi \end{cases}, \qquad f(x+k2\pi)=f(x), \qquad k \in \mathbb{Z}$$

Ildikó Perjési-Hámori

Mathematics A/3

Pécs

2019

The Mathematics A/3 course material was developed under the project EFOP 3.4.3-16-2016-00005 "Innovative university in a modern city: open-minded, value-driven and inclusive approach in a 21st century higher education model".

Ildikó Perjési-Hámori

Mathematics A/3

Pécs

2019

A Mathematics A/3 tananyag az EFOP-3.4.3-16-2016-00005 azonosító számú, „Korszerű egyetem a modern városban: Értékközpontúság, nyitottság és befogadó szemlélet egy 21. századi felsőoktatási modellben" című projekt keretében valósul meg.

Engineering Mathematics 3

Differential equations

PTE MIK Perjésiné dr. Hámori Ildikó

Introduction

Thomas Robert Malthus: An Essay on the Principle of Population (1798)

Problem: The simplest mathematical model of population growth is obtained by assuming that the rate of increase of the population at any time is proportional to the size of the population at that time.

$$\boldsymbol{\frac{dN}{dt} = rN},$$

differential equation

$$N(0) = N_0$$

initial condition

where $N(t)$ is the size of population (dependent variable), t is the time (dependent variable), r is a positive constant.

Question: $\boldsymbol{N(t)} = ?$

unknown: function

Solution: $\frac{1}{N}N' = r$ if $N \neq 0$ $\quad [ln|N(t)|]' = r$

$$\ln(|N(t)| = rt + ln|C| \qquad C \neq 0 \qquad \ln(|N(t)| = lne^{rt} + ln|C| = ln|Ce^{rt}|$$

$$\boldsymbol{N(t) = Ce^{rt}}$$

general solution

If $N = 0$ then $N' = 0$, then $N' = rN$ is true. So

$$\boldsymbol{N = 0}$$

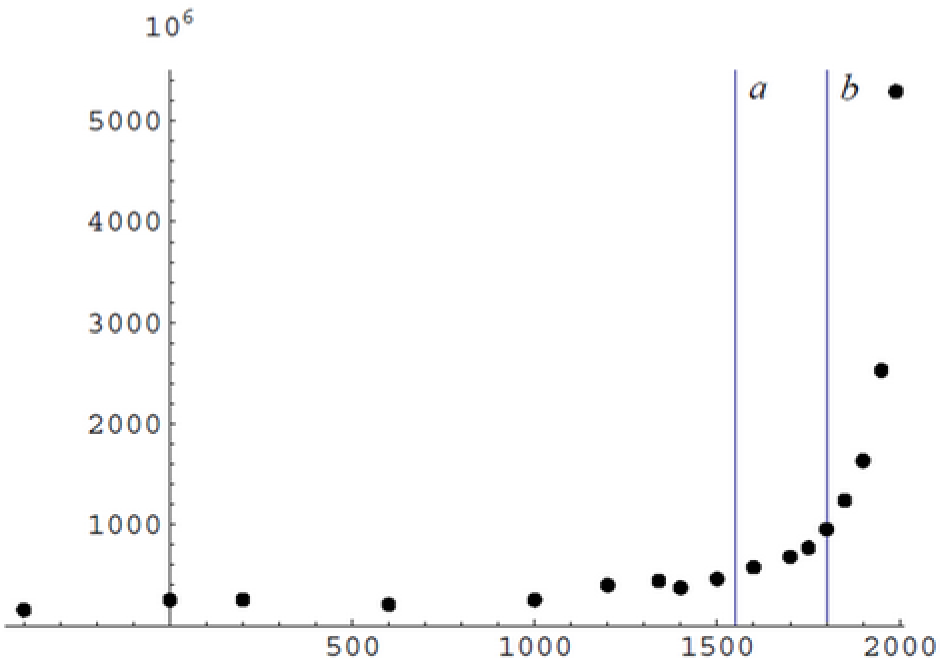
singular solution

Substituting the initial condition: $N_0 = Ce^{r0} = C$

$$\boldsymbol{N(t) = N_0 e^{rt}}$$

particular solution

Introduction



The population of Earth between 400-2000

Concept and classification of ODE

Definítion: A differential equation is an equation that relates some unknown function with its derivatives.

Comment: In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two.

Definítion: An ordinary differential equation (*ODE*) is an equation containing a function of one independent variable and its derivatives.

Classification:

Order the highest derivative order that appears in the equation.

first order: $f(y',y,x)=0$ e.g. $y'x+2\sin x=\cos y$

second order: $f(y'',y',y,x)=0$ e.g. $y'''+y=2e^{x}$

Linearity: **linear** if the unknown function's and its derivatives *degree* is only 1 and products of the unknown function and its derivatives are not allowed. $y'+y\cdot tg\,x=3$

non linear: $ln\,y+2y'=3$ $y\cdot y'+2=x$

Homogeneity: **homogeneous** if each term consists of the unkown function or its derivatives

$$y'\cdot y+2yx=0$$

$$y'\cdot y+2yx=cos\,x$$

inhomogeneous otherwise

Types of solutions of ODE

Def: A function is a **solution** of the differential equation if the equation is satisfied for all values of the independent variable in the domain of the function.

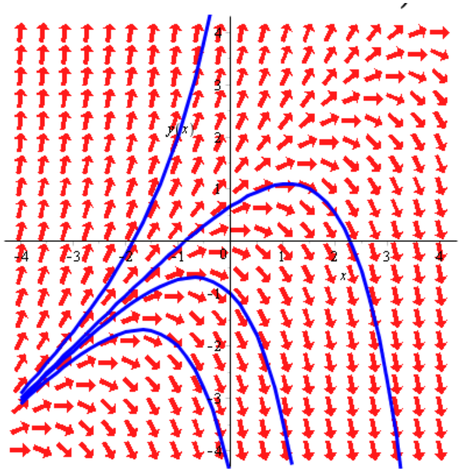
Def: **Initial Condition(s)** are a condition, or a set of conditions, on the solution that will allow us to determine which solution we are after.

Def: ***General solution:*** is a solution that contains all possible solutions

Def: ***Particular solution*:** a solution that satisfies the initial condition

Def: ***Singular solution***, one that cannot be obtained from the general solution. It appears in differential equations when there is a need to divide a term that might be equal to zero

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves.



$$de := \frac{d}{dx}y(x) = y(x) - x \quad gensol := y(x) = 1 + x + e^x\_C1$$

$$partsol := y(x) = 1 + x - \frac{1}{3}e^x \; y(0) = \frac{2}{3}$$

$$y(0) = \frac{2}{3}, y(0) = -1, y(-1) = 2, y(-1) = -2$$

Direction field of ODE

The method of **directional fields** is a graphical method for displaying the general shape and behavior of solution to $y' = f(y, x)$. It soes not require solving differential equation. A direction field line segment is represented by an arrow, to show the direction of the tangent.

The tangent vector for $\boldsymbol{r} = x\boldsymbol{i} + y(x)\boldsymbol{j}$ are drawn from $\boldsymbol{r}' = x\boldsymbol{i} + f(x, y)\boldsymbol{j}$. Each time we specify an initial condition $y(x_0) = y_0$ for the solution $y' = f(y, x)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there.

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$$gensol := y(x) = 1 + x + e^x \_C1$$

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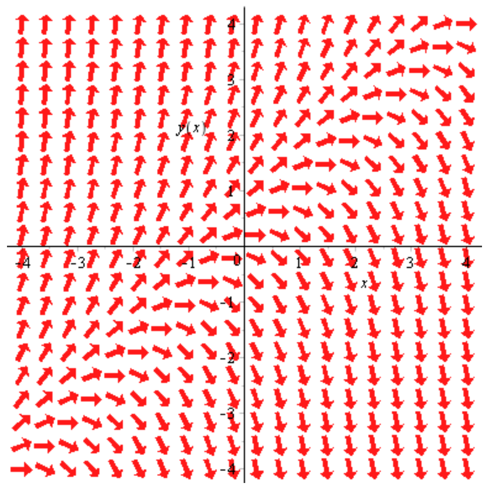
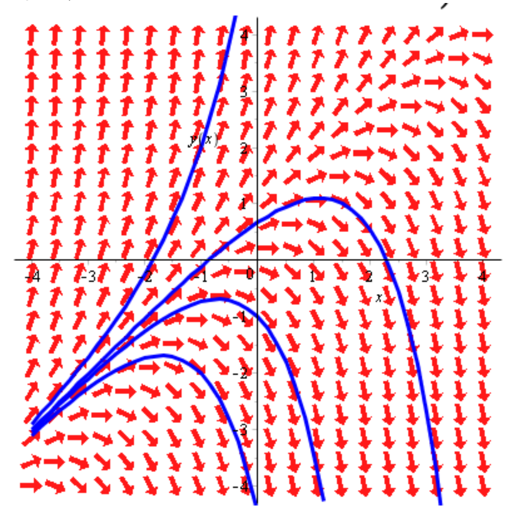


direction field



direction field and particular solutions

$$y(0) = \frac{2}{3}, y(0) = -1, y(-1) = 2, y(-1) = -2$$

:

First order ODE

$\boxed{F(x,y,y')=0}$ implicit form $\boxed{y'=f(x,y)}$ explicit form

Separable differential equation

$$\boxed{y'=f(x)\cdot g(y)}$$

Solution: If $g(y)\neq 0$

$$\frac{y'}{g(y)}=f(x)$$

Integration of both sides respect to *x* (independent variable)

$$\int \frac{y'}{g(y)}dx=\int f(x)dx$$

Knowing that l $y=y(x)$ $y'dx=dy$

$$\int \frac{1}{g(y)}dy=\int f(x)dx$$

$$\boxed{G(y)=F(x)+c}$$

If $y(x)$ can be expressed, we get the solution in explicit form, if not, the solution is in implicit form.

Application

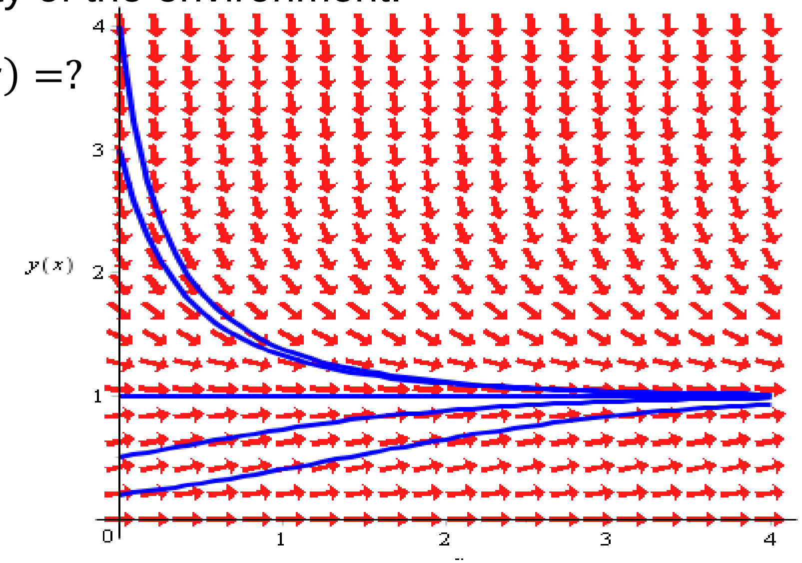
Verhulst equation: when the population growth is constrained by limited resources, a heuristic modification of the Malthusian growth model results in the Verhulst equation

(If the the population increases, several factors will begin to affect the growth rate. For example, there will be increased competition for the limited resources that are available, increases in disease, and overcrowding of the limited available space, all of which would serve to slow the growth rate.)

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right)$$

where r is the growth rate, k is the carrying capacity of the environment.

Let $\frac{N}{k} = y, \quad rt = x.$ Then $\quad y' = y(1-y) \quad y(x) = ?$



General solution: $y(x) = \frac{1}{1+Ce^{-x}} \quad C \neq 0$

Singular solutions: $y = 0$ (no population)

$y = 1$ (k=N)

Initial condition: $y(0) = y_0$,

Particular solution: $y(x) = \frac{y_0}{y_0+(1-y_0)e^{-x}}$

$$\lim_{x\to\infty} y(x) = 1$$

The population, therefore, grows in size until it reaches the carrying capacity of its environment.

:

First order ODE

Linear, inhomogeneous differential equation

$$\boxed{y' + p(x)y = q(x)}$$

If $q(x)=0$ $\quad y' + p(x)\cdot y = 0$ equation is homogenous

Solution: : Variable coefficient method

First solve the homogeneous ODE. $Y' + p(x)Y = 0$ (to distinguish *Y* was written.).

This is a separable ODE

$$\frac{Y'}{Y} = -p(x) \qquad \int \frac{1}{Y}dY = -\int p(x)dx$$

$$ln|Y| = -\int p(x)dx + ln|C| \qquad C \neq 0 \qquad Y = Ce^{-\int p(x)dx}$$

Using this solution to find the solution of the inhomogeneous ODE, substitute *C* constant with *C(x)* unknown function.

$$y = C(x)e^{-\int p(x)dx}$$

$$y' = C'(x)e^{-\int p(x)dx} - C(x)p(x)e^{-\int p(x)dx}$$

Our goal is to determine *C(x).*

Substitue y and y' back to the original ODE.

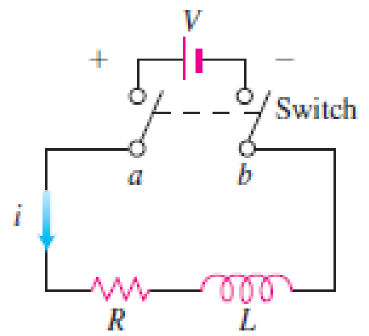
$$C'(x)e^{-\int p(x)dx} - C(x)\cdot p(x)\cdot e^{-\int p(x)dx} + p(x)\cdot C(x)e^{-\int p(x)dx} = q(x)$$

$$C'(x)e^{-\int p(x)dx} = q(x) \qquad C'(x) = q(x)e^{\int p(x)dx} \qquad C(x) = \int q(x)e^{\int p(x)dx}dx$$

$$\boxed{y = \left[\int q(x)e^{\int p(x)dx}dx\right]\cdot e^{-\int p(x)dx}}$$

Application

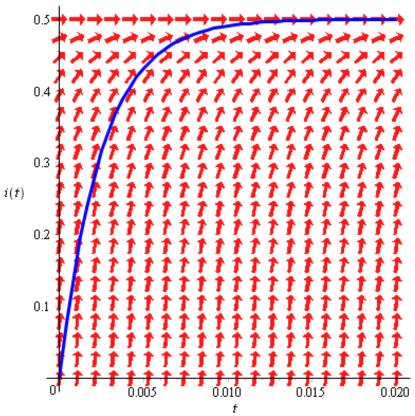
RL circuit The figure represents an electrical circuit whose total resistance is a constant *R* ohms and whose self-inductance, shown as a coil, is *L* henries, also a constant. There is a switch whose terminals at *a* and *b* can be closed to connect a constant electrical source of *V* volts.



Close: The equation accounting for both resistance and inductance is

$$L\frac{di}{dt} + Ri = V, \quad i(0) = 0$$

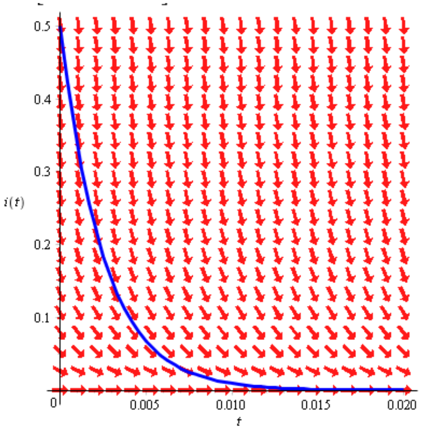
$$i(t) = \frac{V}{R} - \frac{e^{-\frac{Rt}{L}} V}{R}$$



Open:

$$L\left(\frac{d}{dt} i(t)\right) + i(t) R = 0$$

$$i(t) = \frac{V e^{-\frac{Rt}{L}}}{R}$$



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Numerical solution of first order ODE – Euler method

If we do not require or cannot find an *exact* solution giving an explicit formula for an initial value problem $y' = f(x,y), y(x_0) = y_0$, we can generate a table of approximate numerical values of *y* for values *x* in an approximate interval. This type of the solution is called **numerical solution** of the problem.

$$y'(x) = f\big(x, y(x)\big), \qquad y(x_0) = y_0$$

approximate the derivative with the difference quotient

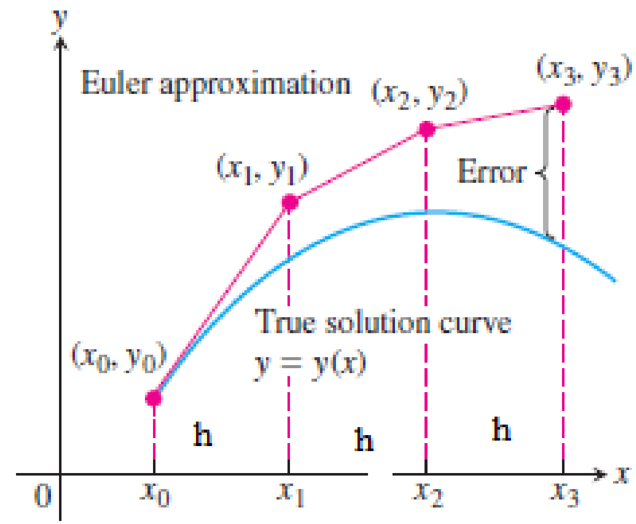
$$y'(x) \approx \frac{y(x_0+h) - y(x_0)}{h} \text{ so}$$

$$\frac{y(x_0+h) - y(x_0)}{h} \approx f\big(x_0, y(x_0)\big)$$

$$y(x_0+h) \approx y(x_0) + hf\big(x_0, y(x_0)\big)$$

$$y(x_0+2h) \approx y(x_0+h) + hf\big(x_0+h, y(x_0+h)\big)$$

$$\vdots$$



$$y(x_0+nh) \approx y(x_0+(n-1)h) + hf\big(x_0+(n-1)h, y(x_0+(n-1)h)\big)$$

:

Second order ODE

Linear, constant coefficients, homogeneous differential equation

$$\boxed{y'' + by' + cy = 0} \qquad b,\ c \in \mathbf{R}$$

Def: $y_1(x)$ and $y_2(x)$ functions are lineary independent of each other, if

$$c_1 y_1(x) + c_2 y_2(x) = 0 \Leftrightarrow c_1 = c_2 = 0 \qquad (c_1, c_2 \in \mathbf{R})$$

Theoreme: If y_1 and y_2 are two independent particular solutions of a linear, constant coefficient(s), homogenous differential equation, then the general solution is :

$$y = c_1 y_1 + c_2 y_2$$

Solution: Find two independent particular solutions

$$y = e^{\lambda x} \qquad y' = \lambda e^{\lambda x} \qquad y'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0 \qquad e^{\lambda x}\left(\lambda^2 + b\lambda + c\right) = 0$$

$$\lambda^2 + b\lambda + c = 0$$

characteristic equation of ODE (quadratic equation to λ-ra)

Depending on the discriminant $D = b^2 - 4c$ there are 3 different cases:

:

Second order ODE

I. $\boxed{D=b^2-4c>0}$ **two different real roots** $\lambda_1, \quad \lambda_2$

$y_1=e^{\lambda_1 x} \quad y_2=e^{\lambda_2 x}$ general solution $\boxed{y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}}$

II. $\boxed{D=b^2-4c=0}$ $\lambda_1=\lambda_2=-\frac{b}{2}$ **one double real root** $y_1=e^{-\frac{b}{2}x}$

It can be proved, that $y_2=xe^{-\frac{b}{2}x}$ is a solution as well, and y_1, y_2 are linearly independent of each other.

general solution $\boxed{y=c_1e^{-\frac{b}{2}x}+c_2xe^{-\frac{b}{2}x}}$

III. $\boxed{D=b^2-4c<0}$ there is **no real solution**

$$\lambda_{1,2}=-\frac{b}{2}\pm\frac{\sqrt{b^2-4c}}{2}=-\frac{b}{2}\pm\underbrace{\sqrt{-1}}_{i}\underbrace{\frac{\sqrt{4c-b^2}}{2}}_{\alpha}$$

complex unit real number

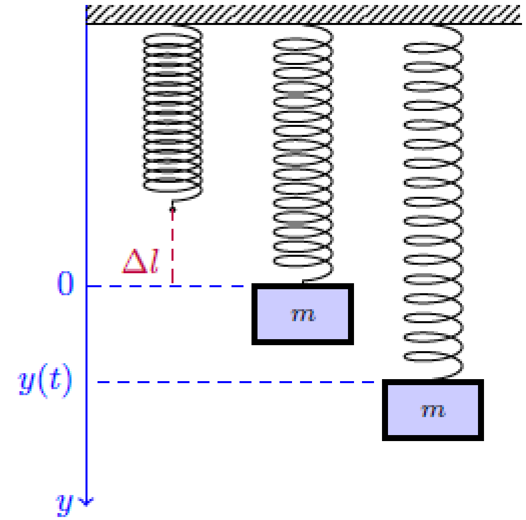
It can be proved, that the two linearly independent solutions are:

$y_1=e^{-\frac{b}{2}x}\cdot\cos\alpha x \quad y_2=e^{-\frac{b}{2}x}\cdot\sin\alpha\, x$ general solution $\boxed{y=e^{-\frac{b}{2}x}\left(c_1\cdot\cos\alpha\, x+c_2\sin\alpha\, x\right)}$

Application

When a spring is slightly deformed, it creates a force proportional and in opposite direction to the deformation. Consider a spring-plus-body system with a spring constant *D*, body mass *m*.

From Newton's law:



$$ma = -Dy$$

$$m\frac{d^2y}{dt^2} = -Dy$$

Let $\frac{D}{m} = \omega^2$, and $\frac{d^2y}{dt^2} = \ddot{y}$ then

$$\ddot{y} + \omega^2 y = 0$$

General solution: $y(t) = \_C1 \sin(\omega t) + \_C2 \cos(\omega t)$

Initial conditions: $y(0) = 0, \dot{y}(0) = vmax, \frac{vmax}{\omega} = A$.

Particular solution: $y(t) = \frac{vmax \sin(\omega t)}{\omega} = A sin(\omega t)$

:

Second order ODE

Linear, constant coefficients, inhomogenous differential equation

$$\boxed{y'' + by' + cy = r(x)} \qquad b, c \in \mathbf{R}$$

Where $r(x)$ is a linear combination of $e^{\alpha x}, sin\beta x, cos\beta x$ and a power function.

Solution: Unknown coefficient method

Steps.

1. Solve the homogeneous equation.
2. Write a function similar to $r(x)$ with unknown coefficients
3. Substitute this function with its derivatives in the original differential equation.
4. Write an equation system, knowing that the coefficients of the same functions are the same on both sides.
5. Solving the equation system get the unknown coefficients and the particular solution of the ODE.
6. The general solution of the original ODE is the sum of the general solution of the homogeneous ODE and the particular solution of the inhomogeneous ODE.

Second order ODE – numerical method

$$y''(x) = f(x, y(x), y'(x)), \qquad y(x_0) = y_0, y'(x_0) = v_0$$

Reduce the ODE into two first order ODE

Notation: $v(x) = y'(x)$, $v'(x) = f(x, y(x), v(x))$

Initial conditions: $y(x_0) = y_0, v(x_0) = v_0$

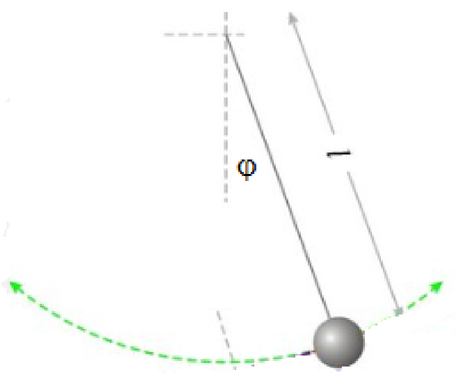
Using the Euler method:

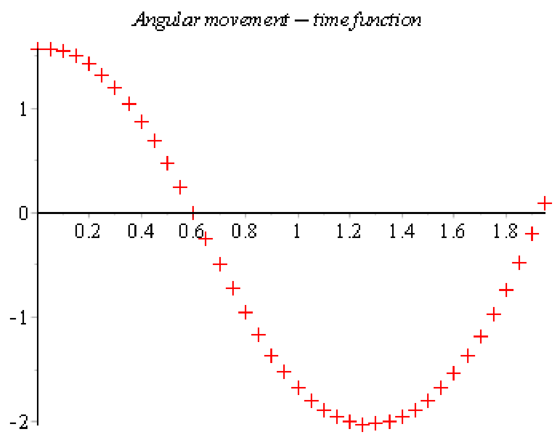
$$y(x_0 + h) \approx y_0 + h\, v(x_0)$$
$$v(x_0 + h) \approx v_0 + h f(x, y(x), v(x))$$
$$\vdots$$

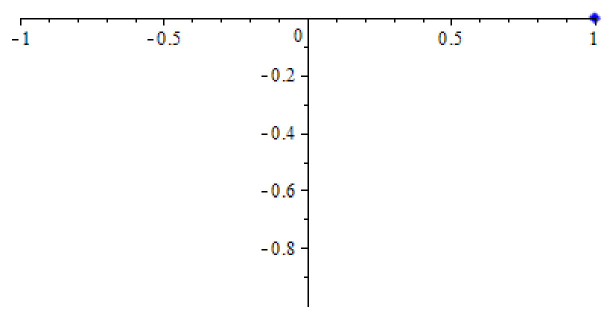
Example:
Pendulum motion, when the initial deflection is greater then 5°

$$\frac{d^2}{dt^2}\varphi(t) = -10\sin(\varphi(t))$$

$$\varphi(0) = \frac{\pi}{2}, D(\varphi)(0) = 0$$







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Laplace transformation

The Laplace transform is most useful for solving **linear, constant-coefficient ODE**'s when the **inhomogeneous term** or its derivative is **discontinuous**.
The main idea is to Laplace transform the constant-coefficient differential equation for $y(t)$ into a simpler algebraic equation for the Laplace-transformed function $Y(s)$ solve this algebraic equation, and then transform $Y(s)$ back into $y(t)$.

Def. Laplace transformation of $f(t)$, denoted by $F(s) = L\{f(t)\}$ is defined by the integral transform

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$

$f(t)$	$F(s)$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

:

Laplace transformation

Heaviside function: $H(t) = \begin{cases} 0 & ha\ t < 0 \\ 1 & ha\ t \geq 0 \end{cases}$ Laplace transform: $F(s) = \frac{1}{s}$

Discontinuous function: $f(t) = \begin{cases} 0 & if\ t < a \\ 1\ \ ha & a \leq t < b \\ 0 & if\ t \geq b \end{cases}$

Laplace transform: $F(s) = \frac{e^{-bs} - e^{-as}}{s}$

Using Heaviside function: $f(t) = H(t-a) - H(t-b)$

The Dirac delta function denoted as $\delta(t)$ is defined by requiring that for any function $f(t)$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

with other words $\delta(t) = \begin{cases} 0 & ha\ t \neq 0 \\ \infty & ha\ t = 0 \end{cases}$

Remark: $\frac{dH(t)}{dt} = \delta(t)$

:

Laplace transforms

Notation: $\frac{d}{dt}f(t) = \dot{f}(t), \frac{d^2}{dt^2}f(t) = \ddot{f}(t)$ (If t is the time in physical applications)

Theorem1 Suppose $\lim_{t\to\infty} e^{-st}f(t) = 0$ and that $f(t)$ is continuous and $\dot{f}(t)$ is piecewise continuous on any interval $0 \le t \le A$, then

$$L\{\dot{f}(t)\} = sL\{f(\mathrm{t})\} - f(0)$$

Proof

$$L\{\dot{f}(t)\} = \int_0^\infty \dot{f}(t)e^{-st}dt = [e^{-st}f(t)]_0^\infty - \int_0^\infty -s\,e^{-st}f(t)dt = -f(0) + sL\{f(\mathrm{t})\}$$

Theorem2 Suppose $\lim_{t\to\infty} e^{-st}f(t) = 0$ and that $\dot{f}(t)$ is continuous and $\ddot{f}t)$ is piecewise continuous on any interval $0 \le t \le A$, then

$$L\left\{\ddot{f}(\mathrm{t})\right\} = s^2L\{f(\mathrm{t})\} - sf(0) - \dot{f}(0)$$

Def. The inverse Laplace transform, denoted L^{-1} , of a function *F* is

$$L^{-1}\{F(s)\} = f(t) \Leftrightarrow \mathrm{F(s)} = \mathrm{L}\{f(t)\}$$

:

Solution linear ODEs using Laplace transformation

Let the ODE' s shape

$$a\ddot{y}(t) + by(t) + y(t) = r(t), \qquad \dot{y}(0) = y_0, \quad \cdot\; y(0) = v_0$$

The steps of the solution:

1. Take the Laplace transforms of both sides, using the Theorem 1 and 2

$$\mathrm{a}\left(s^2F(s) - sy(0) - \dot{y}(0)\right) + \mathrm{b}\left(\mathrm{sF(s) - y(0)}\right) + \mathrm{F(s)} = L\{r(t)\}$$

2. Solve this equation to $F(s)$

3. Take the inverse Laplace transform to $F(s)$. It is the particular soltion of the original ODE.

Engineering Mathematics 3

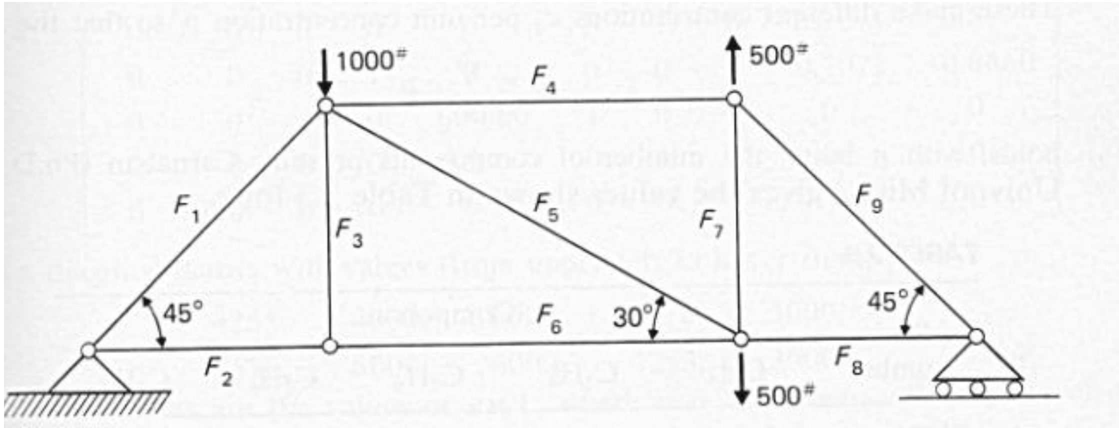
Linear Algebra

What is it about?

The fundamental problem of linear algebra is to solve *n* linear equations in *m* unknowns

Example from mechanics

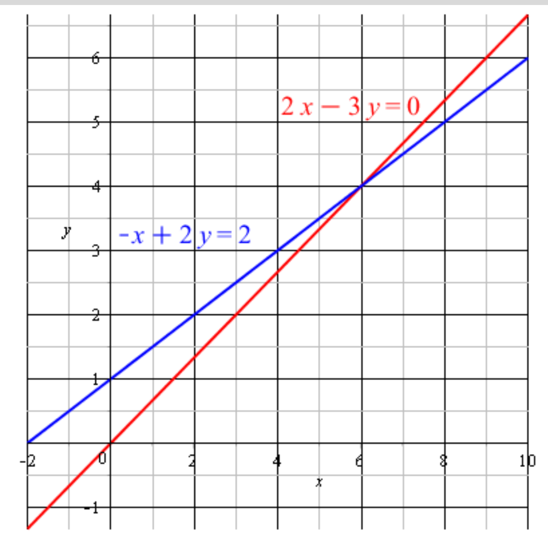
$$\cos\left(\frac{\pi}{4}\right)F_1 - F_4 = 0:$$
$$\sin\left(\frac{\pi}{4}\right)F_1 + F_3 + \sin\left(\frac{\pi}{6}\right)F_5 = -1000:$$
$$F_2 - F_6 = 0:$$
$$-F_3 = 0:$$
$$F_7 + \sin\left(\frac{\pi}{4}\right)F_9 = 500:$$
$$F_4 - \cos\left(\frac{\pi}{4}\right)F_9 = 0:$$
$$\cos\left(\frac{\pi}{6}\right)F_5 + F_6 - F_8 = 0:$$
$$-\sin\left(\frac{\pi}{6}\right)F_5 - F_7 = -500:$$
$$F_8 + \cos\left(\frac{\pi}{4}\right)F_9 = 0:$$



Forms of a linear equation system

$$2x - 3y = 0$$
$$-x + 2y = 2$$

Row form:



The intersection of the plots (if they do intersect) represent the solution to the system of equations.

$x = 6, y = 4$

Vector form:

$$x\begin{bmatrix}2\\-1\end{bmatrix} + y\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

$\mathbf{c} = \begin{bmatrix}2\\-1\end{bmatrix}$, $\mathbf{d} = \begin{bmatrix}-3\\2\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}0\\2\end{bmatrix}$

are vectors,

x, y are scalars.

$x\mathbf{c} + y\mathbf{d}$ is the linear combination of vectors **c** and **d**

$\boxed{x\mathbf{c}+y\mathbf{d}=\mathbf{b} \text{ is the vector form}}$

If $x = 6, y = 4$, then

$$6\begin{bmatrix}2\\-1\end{bmatrix} + 4\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

Matrix form:

$$\begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

$A = \begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}$

is called a coefficient matrix

$\mathbf{x} = \begin{bmatrix}x\\y\end{bmatrix}$ is an unknown vector,

$\boxed{A\mathbf{x} = \mathbf{b} \text{ is the matrix form}}$

If $\mathbf{x} = \begin{bmatrix}6\\4\end{bmatrix}$ then

$$\begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}\begin{bmatrix}6\\4\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

Matrices and matrix operations

Definítion: A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix

$$A = A_{mxn} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$ Size is m by n (written $m \times n$ or mn). In a size description, the first number denotes the number of rows, and the second denotes the number of columns.

A matrix with only one column is called a **column vector** or a **column matrix**, and a matrix with only one row is called a **row vector** or a **row matrix**

$$\mathbf{a} = \bar{a} = [a_1, a_2 \ldots \ldots .., a_n]$$

$$\mathbf{b} = \bar{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

A matrix A with n rows and n columns is called a **square matrix** of order n, the $a_{11}, a_{22} \ldots \ldots .., a_{nn}$ entries are the **main diagonal** of A.

A square matrix is called **identity matrix**, if the entries of the main diagonal are 1, the other entries are 0.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is any matrix, then the **transpose** of A, denoted by A^T, is defined to be the matrix that results by interchanging the rows and columns of A.

Matrices and matrix operations

Definítion: Two matrices are **equal** if they have the same size and their corresponding entries are equal. $A_{mxn}=B_{mxn}$ if $a_{ik}=b_{ik}$ $\forall$ i,, k (i=1…m, k=1…n)

Definition: If A and B are matrices of the same size, then the **sum** $A+B$ (**difference** A-B)

$$\lfloor a_{ik} \rfloor_{m,n} \pm \lfloor b_{ik} \rfloor_{m,n} = \lfloor a_{ik} \pm b_{ik} \rfloor_{m,n}$$

Properties: $A+B=B+A$ commutative

$(A+B)+C=A+(B+C)$ associative

$A+0=A$ zero element (a matrix for which every entry is 0)

$A+X=0$ (X=-A)

Definítion: If A is any matrix and λ is any scalar, then the product λA is the matrix obtained by multiplying each entry of the matrix A by λ. The matrix λA is said to be **a scalar multiple** of A

$$\lambda[a_{ik}]=[\lambda a_{ik}]$$

Properties:: $(\lambda_1\lambda_2)A=\lambda_1(\lambda_2 A)$ associative

$(\lambda_1+\lambda_2)A=\lambda_1 A+\lambda_2 A$ distributive

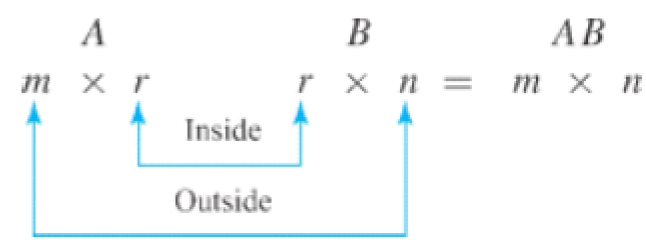
$\lambda(A+B)=\lambda A+\lambda B$ distributive

Multiplying Matrices

$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ because $2 \cdot 6 + (-3) \cdot 4 = 0$, and $(-1) \cdot 6 + 2 \cdot 4 = 2$

Definition: $AB = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mr} \end{bmatrix}\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{r1} & \cdots & b_{rn} \end{bmatrix}$ the entry in row i and column j of AB is given by $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ir}b_{rj}$.

Remark: The definition of matrix multiplication requires that the number of columns of the first factor *A* be the same as the number of rows of the second factor *B* in order to form the product *AB*.



Properties:

$AB \neq BA$ non commutative

$(AB)C = A(BC)$ associative

$(A+B)C = AB + AC$ distributive

Determinant

Definition: The determinant of the matrix A_{2x2} is the following number:

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Definition: If A is a square matrix then the minor of entry a_{ij} is denoted A_{ij} and is defined to be the determinant of the submatrix that remains after the i_{th} row and j_{th} column are deleted from A. The number $C_{ij} = (-1)^{i+j} A_{ij}$ is called the **cofactor** of entry a_{ij}

Definition: If A is a nxn matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactor and adding the resulting product is called the **determinant** of A, and the sums themselves are called cofactor expansions of A.

$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$ (cofactor expansion along the jth column)

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$ (cofactor expansion along the ith row)

Remark: Note that a minor and its corresponding cofactor are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the "checkerboard" array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Determinant

Definition: If det $A \neq 0$ then A is called **regular matrix**, if det A =0 then a is called **singular matri**x.

Rule of Sarrus: Determinants of 3x3 matrices can be evaluated very efficiently using the pattern suggested in the next figure:

$$\det A = \left|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{12}a_{21}a_{33})$$

Properties: Let A be a square matrix,

1. if A has a row of zeros or a column of zeros, then det$A$=0.
2. det A=det A^T
3. if B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then det B=$k$det A.
4. If B is the matrix that results when two rows or two columns of A are interchanged then det B= −det A.
5. If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column then det B=det A.

Adjoint and inverse of a matrix

Definition If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij} then the matrix

$$adjA = \begin{bmatrix} C_{11} & C_{21} & & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ & & & \\ C_{1n} & & & C_{nn} \end{bmatrix}$$

is called the **adjoint** of A.

Remark: adj$A$ is the transpose of the cofactor matrix.

Theoreme: If A is any nxn matrix (adj A)A=A(adj A)=(det$A$)I, where I is the identity matri**x.**

Corrolary: If det $A \neq 0$, then $$\frac{A(adjA)}{\det A} = I$$

Definition: If A is a regular square matrix (det $A \neq 0$), A^{-1} is called the inverse of the matrix, if $A^{-1}A = AA^{-1} = I$, where I is the identity matrix.

Corollary: $$A^{-1} = \frac{adjA}{\det A}$$

Solution of a regular linear equation system

Defintion The form of a linear equation system with as many equations as unknowns (n) is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_{n1}$$

where the x_k ($k = 1,2,\ldots,n$) is the kth unknown, the a_{ik} ($i = 1,2, \ldots ,n$; $k = 1,2, \ldots,n$) , b_i are real numbers.

Define a matrix from the coefficients, from the unknown and from the constants from the right hand side.

$$A = A_{nxn} = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ & & \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$$

The short form of the system is $A\mathbf{x}=\mathbf{b}$.

Definition: A linear equation system is called a **regular** one, if it has as many equations as unknowns and det $A \neq 0$.

Cramer's rule

Cramer's rule: The solution of a regular equation system is $\mathbf{x}=A^{-1}\mathbf{b}$

Theorem: If $A\mathbf{x}=\mathbf{b}$ is a system of n linear equations in n unknown such det $A \neq 0$, then the system

- is solvable
- has a unique solution.

The solution is $x_1 = \frac{\det A_1}{\det A}$, $x_2 = \frac{\det A_2}{\det A}$,..., $x_n = \frac{\det A_n}{\det A}$ where A_j is the matrix obtained by replacing the entries in jth column of A in matrix $\mathbf{b}$.

Matrix form of a general linear equation system

Definition: A system of *m* linear equations in *n* unknown is

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots + a_{1k}x_k + \dots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \dots + a_{2k}x_k + \dots + a_{2n}x_n = b_2 \\
&. \\
&a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ik}x_k + \dots + a_{in}x_n = b_i \\
&. \\
&a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mk}x_k + \dots + a_{mn}x_n = b_m
\end{aligned}
$$

where the x_k ($k = 1,2,\dots,n$) is the *k*th unknown, the a_{ik} ($i = 1,2,\dots,m$; $k = 1,2,\dots,n$), b_i are real numbers.

$$
A = A_{mxn} = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ & & \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}
\quad
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}
\quad
\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}
\quad
B = B_{mx(n+1)} = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} & b_1 \\ a_{21} & a_{22} & . & . & a_{2n} & b_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} & b_n \end{bmatrix}
$$

coefficient matrix — unknown vector — constant vector — augmented matrix

$A\mathbf{x}=\mathbf{b}$

Existence and uniqueness of the solution, elementary row operations

A **solution** of the system is a list $\{s_1, s_2, \dots, s_n\}$ of numbers that makes each equation a true statement when the values $s_1, s_2, \dots, s_n$ are substituted for $x_1, x_2, \dots, x_n$, respectively.

A system of linear equations has
- no solution or
- exactly one solution or
- infinitely many solutions

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Questions:

1. Is the system consistent that is does at least one solution *exists*?
2. If a solution exists, is it the only one; that is, is the solution *unique*?

Algebraic operations that do not alter the solution set:
E1. Multiply an equation through by a nonzero constant.
E2. Interchange two equations.
E3. Add a constant times one equation to another.

Row operations with an augmented matrix correspond to the equation:
M1. Multiply a row through by a nonzero constant.
M2. Interchange two rows.
M3. Add a constant times one row to another.

Reduced row echelon form: Gauss-Jordan elimination

Forward phase: A matrix is in **row echelon form** if it has the following properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it,
3. All entries in a column below a leading entry are zeros.

Backward phase: In **reduced echelon form** there are two additional conditions:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

row echelon form REF

reduced row echelon form RREF

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

A linear system is **consistent** if and only if the rightmost column of the augmented matrix is *not* a pivot column – that is, if and only if an echelon form of the augmented matrix has *no* row of the form $[0 \ \ldots \ 0 \ b]$ with b nonzero.

Solving a linear system using Gauss-Jordan elimination

1. Write the augmented matrix of the system.
2. Use the reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop, otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equation corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

$$\begin{aligned} x_1+2x_2+3x_3&=1\\ x_1+3x_2+5x_3&=1\\ 3x_1-x_2-4x_3&=1\\ 9x_1+2x_2-x_3&=1\\ 5x_1+2x_2+x_3&=1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1\\ 1 & 3 & 5 & 1\\ 3 & -1 & -4 & 1\\ 9 & 2 & -1 & 1\\ 5 & 2 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & 4\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1&=-1\\ x_2&=4\\ x_3&=-2 \end{aligned}$$

one solution

$$\begin{aligned} x_1+x_2+x_3-x_4&=4\\ x_1-x_2+x_3+x_3&=8\\ 3x_1+x_2+3x_3-x_4&=16 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 4\\ 1 & -1 & 1 & 1 & 8\\ 3 & 1 & 3 & -1 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 6\\ 0 & 1 & 0 & -1 & -2\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1+-8x_2+9x_3&=-32\\ 2x_1-x_2+3x_3&=-1\\ x_1+2x_2-x_3&=12 \end{aligned}$$

$$\begin{bmatrix} 1 & -8 & 9 & -32\\ 2 & -1 & 3 & -1\\ 1 & 2 & -1 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

no solution

$$\begin{aligned} x_1&=6-x_3\\ x_2&=-2+x_4 \end{aligned}$$

infinite number of solutions
with two free variables

Vector space

Definition: Let V be an arbitrary nonempty set of objects, on which the addition and multiplication by scalars are defined. If the following axioms are satisfied by all object **u**, **v**, **w** in V and all scalars λ and μ, then we call V a **vector space** and we call the objects in V *vectors*.

1. If **u** and **w** are objects in V, then **u**+**v** is in V.
2. **u**+**v**= **v**+**u**
3. **u**+(**v**+**w**)=(**u**+**v**)+**w**
4. There is an object 0 in V, called a zero vector for V, such that **0**+**u**= **u**+**0**= **u** .
5. For each **u** in V, there is an object **–u** in V, called a negative of **u**, such that **u**+(**–u**) = (**–u**) +**u**=**0.**
6. If λ is any scalar and **u** is an object in V, then λ**u** is in V.
7. λ(**u**+**v**)=λ**u**+λ**v**
8. (λ+μ)**u**= λ**u**+μ**u**
9. λ(μ**u**)= ($\lambda\mu$)**u**
10. 1**u**=**u**

Remark: The definition of a vector space does not specify the nature of vectors or the operations. Any kind of object can be a vector . The only requirement is that the ten vector space axioms be satisfied.

Examples:

1. Geometrical vectors: $V = \mathbb{R}^2$ (vectors in plane is called 2-space vector), $V = \mathbb{R}^3$ (vectors in space, 3-space)
2. $V = M_{3x3}$ (3 by 3 matrices)
3. V : convergent infinite sequences of real numbers
4. V : real functions with one variable

Subspace, linear combination, linearly independent and dependent set

Definition: A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Subspace examples

1. Planes through the origin are subspaces of $\mathbb{R}^3$
2. $M = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ matrices, $a, b \in \mathbb{R}$ are subspaces of M_{2x2}
3. Number sequence approaches to 0 is a subspace convergent infinite sequences of real numbers

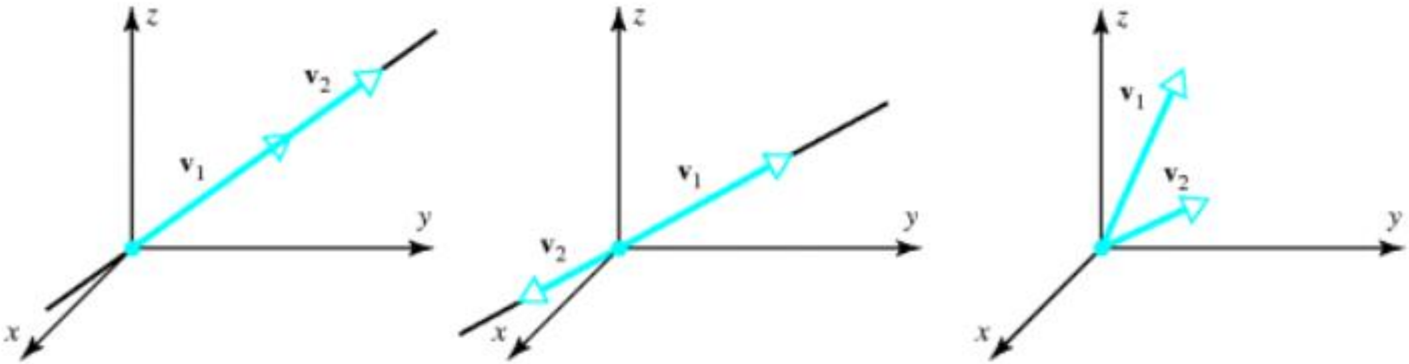
Definition: If **w** is a vector in a vector space V , then **w** is said to be **a linear combination** of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …., $\mathbf{v}_n$ in V, if **w** can be expressed in the form $\mathbf{w} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \ldots + x\lambda_n\mathbf{v}_n$

Definition: Let $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a nonempty set of vectors in a vector space V. If the vector equation $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \ldots + \lambda_n\mathbf{v}_n = \mathbf{0}$ has only **one solution** (trivial solution) namely $\lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_n = 0$, then we call S a **linearly independent set.**
If there are solutions **in addition** to the trivial **solution**, then S is said to be a **linearly dependent set**.

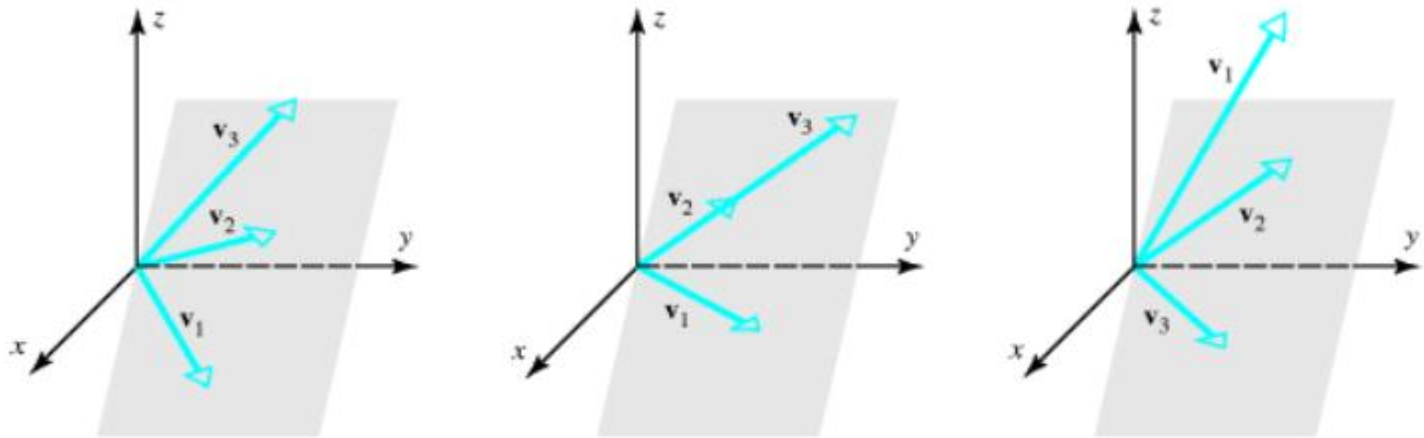
Linearly independent examples

1. In $\mathbb{R}^3$ space $\mathbf{i}$ =(1,0,0), $\mathbf{j}$ =(0,1,0), $\mathbf{k}$ =(0,0,1)
2. In $\mathbb{R}^n$ space the set of standard unit vectors $\mathbf{e_1}$ =(1,0,0,….,0), $\mathbf{e_2}$ =(0,1,0,….,0),… , $\mathbf{e_n}$ =(0,0,0,….,1)

Linearly independent and dependent vectors



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

Span, basis, coordinates, coordinate mapping

Definition: If any **w** vector in a vector space V is expressed as a linear combination

$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ for some coefficients c_i, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are said to **span** the space. Notation: span(V)

Definition: If V is any vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a finite set of vectors in V, then it is called a **basis** for V, if

- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ set is linearly independent
- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ set spans V.

Definition: If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V and

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

is the expression for a vector **w** in terms of the basis S, then scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates** of **w** relative to the basis S, the $\mathbf{w}_S = (c_1, c_2, \ldots, c_n)$ vector is called **coordinate vector** of **w**.

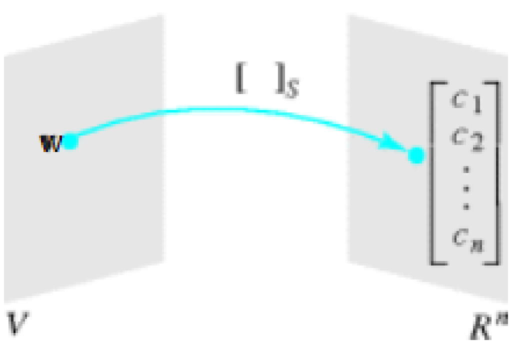
Definition: If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vectorspace V and if $\mathbf{w}_S = (c_1, c_2, \ldots, c_n)$ is the coordinate vector of **w** relative S, then, the **mapping** $\mathbf{w} \rightarrow (\mathbf{w})_S$ creates a one-to-one correspondence between vectors in the *generated* space V and vectors in the *familiar* vector space $\mathbb{R}^n$. It is called a **coordinate map** from V to $\mathbb{R}^n$. The coordinate vector in matrix form:

$$[\mathbf{w}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

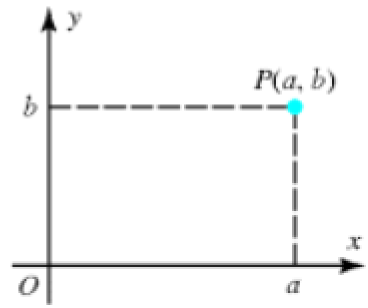
Coordinates in $\mathbb{R}^2$ and $\mathbb{R}^3$

Trivial bases

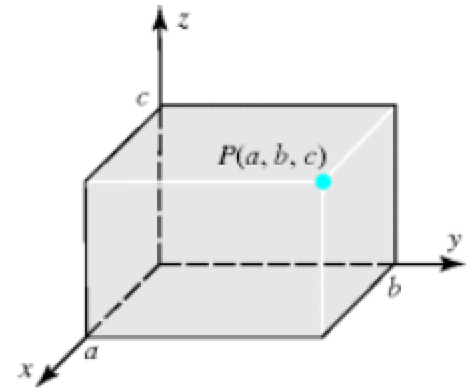
1. In $\mathbb{R}^3$ space $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$
2. In $\mathbb{R}^n$ space the set of standard unit vectors
   $\mathbf{e_1} = (1,0,0,\ldots,0)$,
   $\mathbf{e_2} = (0,1,0,\ldots,0)$,
   $\vdots$
   $\mathbf{e_n} = (0,0,0,\ldots,1)$



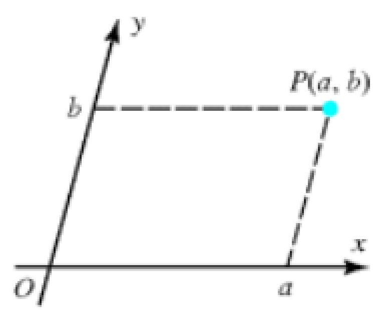
Coordinate map



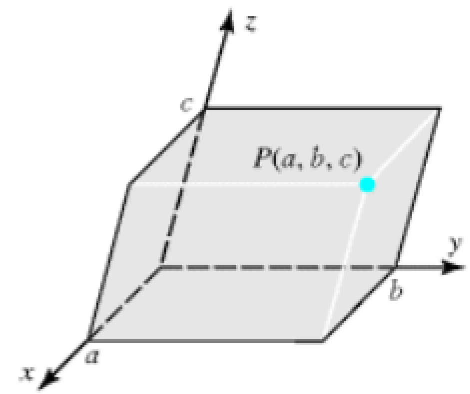
Coordinates of P in a rectangular coordinate system in 2-space.



Coordinates of P in a rectangular coordinate system in 3-space.



Coordinates of P in a nonrectangular coordinate system in 2-space.



Coordinates of P in a nonrectangular coordinate system in 3-space.

Dimension of a vector space, rank of a matrix

Definition: The **dimension** of a finite-dimensional vector space *V* is denoted by dim(V) and is defined to be the number of vectors in a basis for *V*. Dimension in other words: **degrees of freedom.**

For an *nxm* matrix $A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$

Definition:
The vectors
$$\mathbf{v}_1 = [a_{11} \quad a_{12} \quad \dots \quad a_{1n}]$$
$$\mathbf{v}_2 = [a_{21} \quad a_{22} \quad \dots \quad a_{2n}]$$
$$\vdots$$
$$\mathbf{v}_m = [a_{m1} \quad a_{m2} \quad \dots \quad a_{mn}]$$
in $\mathbb{R}^n$ that are formed from the rows of *A* are called the **row vectors of *A***.

Definition:
The vectors $\mathbf{w}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m21} \end{bmatrix}, \dots, \mathbf{w}_1 = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$
in $\mathbb{R}^m$ that are formed from the column of *A* are called the **column vectors of *A***.

Theoreme: The row space and column space of a matrix *A* have the same dimensions

Definition: The common dimension of the row space and column space of a matrix *A* is called the **rank** of *A* and is denoted by rank(*A*);

Change of basis

Problem: If **w** is a vector in a finite-dimensional vector space *V* and if we change the basis from basis *B* to a basis *B*', how are the coordinate vectors $[\mathbf{w}]_B$ and $[\mathbf{w}]_{B'}$.

Definition: Let *B*={$\mathbf{b}_1$, $\mathbf{b}_2$, … $\mathbf{b}_k$,…, $\mathbf{b}_n$ } be a basis of *V*, $\mathbf{c} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots c_k\mathbf{b}_k + \ldots + c_n\mathbf{b}_n$, $c_k \neq 0$ be a vector in *V*.
We call it an **elementary change of basis**, if we change $\mathbf{b}_k$ into **c**, the new basis is *B'*={$\mathbf{b}_1$, $\mathbf{b}_2$, … **c**,…, $\mathbf{b}_n$ }

Procedure:

$\mathbf{c} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_k\mathbf{b}_k + \ldots + c_n\mathbf{b}_n$, $c_k \neq 0$
$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_k\mathbf{b}_k + \ldots + x_n\mathbf{b}_n$

Express $\mathbf{b}_k$ from **c** and substitute it into **x**. Then we get the coordinates of **x** in the new basis.

$\mathbf{b}_k = -\frac{c_1}{c_k}\mathbf{b}_1 - \frac{c_2}{c_k}\mathbf{b}_2 - \ldots + \frac{1}{c_k}\mathbf{c} + \ldots - \frac{c_n}{c_k}\mathbf{b}_n$
$\mathbf{x} = (x_1 - \delta c_1)\mathbf{b}_1 + (x_2 - \delta c_2)\mathbf{b}_2 + \ldots + \delta\mathbf{c} + \ldots + (x_n - \delta c_k)\mathbf{b}_n$
where $\delta = \frac{x_k}{c_k}$

Original basis	**c**	**x**	New basis	**c**	**x**
$\mathbf{b}_1$	c_1	x_1	$\mathbf{b}_1$	0	$x_1 - \delta c_1$
$\mathbf{b}_2$	c_2	x_2	$\mathbf{b}_2$	0	$x_2 - \delta c_2$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
$\mathbf{b}_k$	$\boxed{c_k}$	x_k	**c**	1	δ
.	.	.	.		
.	.	.	.		
.	.	.	.		
$\mathbf{b}_n$	c_n	x_n	$\mathbf{b}_n$	0	$x_n - \delta c_n$

$$\delta = \frac{x_k}{c_k}$$

Change of basis - example

Example: Let $\mathbf{b}_1 = (-1,1,-1), \mathbf{b}_2 = (2,1,0), \mathbf{b}_3 = (1,-1,1)$ be vectors in $\mathbb{R}^3$.

1. Are these vectors linearly dependent or independent vectors?
2. What is the dimension of the space generated by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$?
3. Choose as many basis vectors from $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, as you can.
4. If $\mathbf{c} = (4,3,2)$, $\mathbf{d} = (4,-1,2)$, are these vectors in the space generated by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$?
5. If one of **c** or **d** is in the generated space, give its coordinates in the new basis.

Solution: Create a table from the vectors, where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are vectors of standard basis and use elementary change of basis, as many as possible. We need the coordinate vectors of the old basis relative to the new basis.

	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	c	d
$\mathbf{e}_1$	-1	2	1	4	4
$\mathbf{e}_2$	1	1	-1	3	-1
$\mathbf{e}_3$	-1	0	1	2	2
δ		-2	-1	-4	-4

$\mathbf{e}_1 \Rightarrow \mathbf{b}_1$

	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	c	d
$\mathbf{b}_1$	1	-2	-1	-4	-4
$\mathbf{e}_2$	0	3	0	7	3
$\mathbf{e}_3$	0	-2	0	-2	-2
δ			0	1	1

$\mathbf{e}_3 \Rightarrow \mathbf{b}_2$

	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	c	d
$\mathbf{b}_1$	1	0	-1	-2	-2
$\mathbf{e}_2$	0	0	0	4	0
$\mathbf{b}_2$	0	1	0	1	1

1. $\mathbf{b}_3 = -\mathbf{b}_1$, so e.g. $1\mathbf{b}_1 + 0\mathbf{b}_2 - 1\,\mathbf{b}_3 = \mathbf{0}$ dependent vector system.
2. $\dim(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$=2
3. $B\{\mathbf{b}_1, \mathbf{b}_3\}$
4. **c** is not in the space generated by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ ($\mathbf{c} = -2\mathbf{b}_1 + 4\mathbf{e}_2 + 1\mathbf{b}_3$)
   d is in the space generated by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ ($\mathbf{d} = -2\mathbf{b}_1 + 1\mathbf{b}_3$)
5. $\mathbf{d} = (-2,1)_B$

Vector form of a linear equation system

Definition: A vector form of a linear equation system of *m* linear equations in *n* unknown is

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

where $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ \vdots \\ a_{1m} \end{bmatrix}$ $\mathbf{a}_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ \vdots \\ a_{2m} \end{bmatrix}$ .... $\mathbf{a}_n = \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ \vdots \\ a_{nm} \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$ are vectors.

Alternative questions:

1. Can vector **b** be expressed as a *linear combination* of $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ vectors?
   (Is the system consistent, that is, does at least one solution *exist*?)
2. Are vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ *linearly independent* of each other? (If a solution exists, is it the only one; that is, is the solution *unique*?)

$$\begin{aligned} -x_1 + 2x_2 + x_3 &= 4 \\ x_1 + x_2 - x_3 &= 3 \\ -x_1 + \quad + x_3 &= 2 \end{aligned} \qquad \begin{aligned} 1\cdot x_1 + 0\cdot x_2 + (-1)\cdot x_3 &= -2 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= 4 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 &= 1 \end{aligned}$$

No solution, because of the second row

	x_1	x_2	x_3		
	a_1	a_2	a_3	b_1	b_2
e_1	-1	2	1	4	4
e_2	1	1	-1	3	-1
e_3	-1	0	1	2	2

$$\begin{aligned} -x_1 + 2x_2 + x_3 &= 4 \\ x_1 + x_2 - x_3 &= -1 \\ -x_1 + \quad + x_3 &= 2 \end{aligned} \qquad \begin{aligned} 1\cdot x_1 + 0\cdot x_2 + (-1)\cdot x_3 &= -2 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= 0 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 &= 1 \end{aligned}$$

$x_1 = -2 + x_3$
$x_2 = 1$
Infinite number of solutions with one free parameter

	a_1	a_2	a_3	b_1	b_2
b_1	1	0	-1	-2	-2
e_2	0	0	0	4	0
b_2	0	1	0	1	1

Eigenvalue, eigenvector

Definition: If A is an nxn matrix, then a nonzero $\mathbf{v}$ in $\mathbb{R}^n$ is called an **eigenvector** of A

$$A\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

for some scalar of λ. λ is called an **eigenvalue** of A. and $\mathbf{v}$ is said to be an eigenvector corresponding to λ.

Equivalently :

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

where I is the identity matrix.

If the coefficient $A - \lambda I$ were nonsingular, the unique solution of equation (1) would be $\mathbf{v} = \mathbf{0}$, which is of no interest. To obtain a nonzero solution of the eigenvalue problem, $A - \lambda I$ must be singular.

$$\det(A - \lambda I) = 0$$

This equation is called a **characteristic equation** of A.

Example: If $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$, then the characteristic equation is $\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$ or

the characteristic polynomial is $(-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1) = 0$

So the eigenvalues are $\lambda_1 = -6\,, \lambda_2 = -1$.

If $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $\lambda_1 = -6$ $\quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 + 2x_2 = 0$ $\quad x_1 = -2x_2$ $\quad$ *e.g.* $\mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$2x_1 + 4x_2 = 0$

for $\lambda_1 = -1$ $\quad \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -4x_1 + 2x_2 = 0$ $\quad 2x_1 = x_2$ $\quad$ *e.g.* $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$2x_1 - x_2 = 0$

Diagonization

Properties:

1. The eigenvalues of an upper or lower trigonal matrix are the diagonal entries.
2. The eigenvalues of a diagonal matrix (all the entries except for the main diagonal are zero) are the diagonal entries, and the eigenvectors are the trivial basis of the column vector space.
3. The **eigenvectors** of a real, **symmetric matrix** with distinct eigenvalues are mutually **orthogonal**.
4. The product of eigenvalues equals to the determinant of the matrix.

Definition: A square matrix A is said to be diagonalizable if it is similar to some diagonal matrix, that is, if there exists an invertible matrix P such that $D = P^{-1}AP$is diagonal.

Remark: If $D = P^{-1}AP$ then $A = PDP^{-1}$.

Theorem: An A nxn square matrix is diagonalizable if and only if A has n linearly independent eigenvectors and P is the matrix constructed from the linearly independent eigenvectors.

Example: Show that $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ is a diagonizable, $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$is not diagonizable matrix.

Solution: For A $\lambda_1 = 5, \lambda_2 = 4, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$=$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

For B there is only one eigenvalue $\lambda_1 = 2$ and for all eigenvectors the second coordinate is equal to 0. It has no two independent eigenvectors, so it is not a diagonizable matrix.

Principal axes theorem

Definition: If A is a symmetric nxn matrix, and $\mathbf{x}$ is a nonzero column vector in $\mathbb{R}^n$ then the number

$$Q\big(A(\mathbf{x})\big) = \mathbf{x}^T A\mathbf{x}$$

is called the **quadratic form** associated with A.

If n=2, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$, then $Q = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2$ (in geometry conic sections)

If n=3, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$, then

$Q = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$ (in geometry second ordered surfaces)

The numbers x_1, x_2, x_3 are the coordinates in the $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$trivial basis. Let the eigenvalues of A_{3x3} symmetrical matrix be $\lambda_1, \lambda_2, \lambda_3$ and let the appropriate eigenvectors be $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Make a basis transformation, and let the new basis be the eigenvectors be $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
In the new coordinate system $\mathbf{x}^T = [y_1, \; y_2, y_3] = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3$,

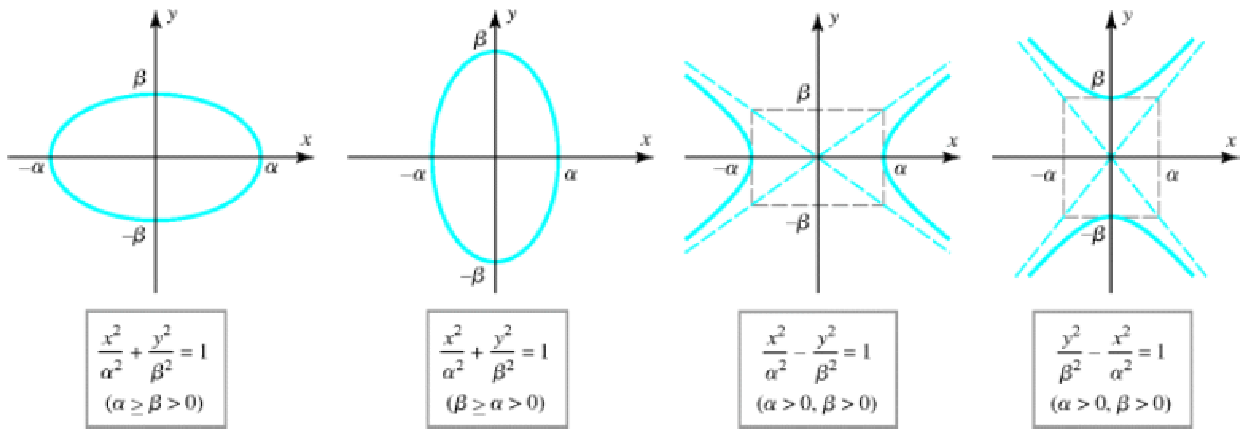
$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \; A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \; A\mathbf{v}_3 = \lambda_3\mathbf{v}_3.$$

Then $\mathbf{x}^T A\mathbf{x} = (y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3)(y_1\lambda_1\mathbf{v}_1 + y_2\lambda_2\mathbf{v}_2 + y_3\lambda_{31}\mathbf{v}_3) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$.

We used $\mathbf{v}_i\mathbf{v}_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$

Definition: We call a basis transformation a **principal axes transformation**, if we transform a quadratic form into the basis of eigenvectors be $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \ldots, \mathbf{v}_n$.

Principal axes theorem



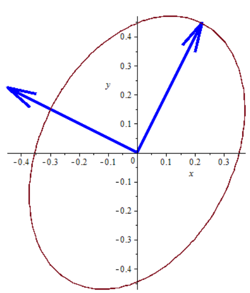
Central conics in standard position:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Central quadratics in standard position:

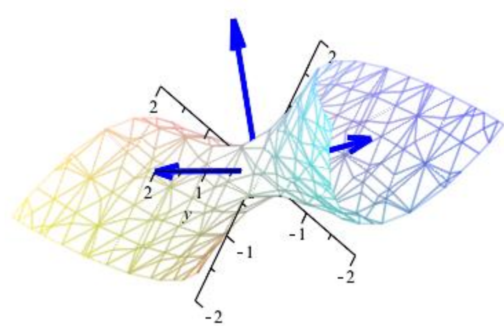
$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

A central conic in not standard form



$$8x^2+5y^2-4xy=1$$

A central quadratic in not standard form



$$2x^2+6y^2+2z^2+8xz=1$$

First-order, linear, homogenous differential equation systems

Definition: Equation $$\mathbf{y}' = A\mathbf{y}$$

where $$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}, A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

is called a **first order, linear, homogenous differential equation system**.

Solution:

- Find a matrix P that diagonalizes A.
- Make a substitution $\mathbf{y} = P\mathbf{u}$, $\mathbf{y}' = P\mathbf{u}'$ to obtain a new „diagonal system” $\mathbf{u}' = D\mathbf{u}$, where $D = P^{-1}AP$
- Solve $\mathbf{u}' = D\mathbf{u}$
- Determine y from the equation $\mathbf{y} = P\mathbf{u}$.

Example: Solve the system $y_1' = y_1 + y_2$
$y_2' = 4y_1 - 2y_2$.
Find the solution that satisfies the initial conditions $y_1(0) = 1, y_2(0) = 6$.

Solution: The coefficient matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$,

$\lambda_1 = 2, \lambda_2 = -3$, $P = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.

$u_1' = 2u_1$, $u_2' = -3u_2$

$\mathbf{u} = \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{-3x} \end{bmatrix}$, $\mathbf{y} = P\mathbf{u}$ then

$$\mathbf{y} = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{-3x} \end{bmatrix} = \begin{bmatrix} c_1 e^{2x} + c_2 e^{-3x} \\ c_1 e^{2x} - 4c_2 e^{-3x} \end{bmatrix}.$$

From the initial conditions: $c_1 + c_2 = 1$
$c_1 - 4c_2 = 6$

$c_1 = 2$, $c_2 = -1$. So $y_1 = 2e^{2x} - e^{-3x}$
$y_2 = 2e^{2x} + 4e^{-3x}$

Mathematics 3

Differential equations1 First order- Analitical and numerical solutions

First order separable ODE

1. The number of bacteria in a certain culture grows at a rate that is proportional to the number present. If the number increased from 500 to 2000 in 2 hours, determine
   a. the number present after 12 hours
   b. the doubling time.
2. In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes.
   $$m\frac{dv}{dt} = -kv \qquad v(0) = v_0 \quad k > 0$$
   What can we learn from the solution of the ODE?
3. A body with mass m free falling under gravity but with air resistance. We assume that the force of air resistance is proportional to the speed of the mass and opposes the direction of motion.
   $$m\frac{dv}{dt} = -mg - kv \qquad v(0) = 200\ \frac{km}{h}$$
   As an example, a skydiver of mass m = 100 kg with his parachute closed may have a terminal velocity of 200 km/h, g $= 9.81\ \frac{m}{s^2}$, $k = 63.54\ \frac{kg}{h}$. What is the speed after $t = 4$ sec?
4. Give the general solution
   a. $(2x+1)y' - 3y = 0$
   b. $(2xy+3y)y' - y^2 - 2 = 0$
   c. $2(xy + x - y - 1) = (x^2 - 2x)y'$
   d. $(x^2 - 1)y' = y$
5. Find the particular solution
   a. $x(1+y) = y'(1+x) \qquad y(0) = 2$
   b. $x\sqrt{1-y^2} = yy'\sqrt{1-x^2} \quad y\left(\frac{1}{2}\right) = \frac{1}{2}$
   c. $(2x+1)y' = 2y(lny + 1) \quad y(1) = e$

First order linear ODE

6. Give the general solution
   a. $y' - 2y = 5$
   b. $y' + 3y = e^{2x}$
   c. $y'cosx + ysinx = 1$
   d. $xy' - 2y = x^3 + 1$
   e. $xy' + y = xlnx$
7. Find the particular solution
   a. $y' - 2y = 5 \qquad y(1) = 3$
   b. $y' + 3x^2y = x^2 \qquad y(1) = \frac{4}{3}$
   c. $y' + y = e^{-x} \qquad y\left(ln\frac{1}{2}\right) = 2$
   d. $y' + 2xy = 2xe^{-x^2} \quad y\left(\sqrt{ln2}\right) = \frac{1}{2}(1 + ln2)$
8. Find the analitical and numerical solutions
   a. $y' = 2xy \qquad y(0) = 1 \quad a = 0 \quad b = 1 \quad n = 5$
   b. $y' = y - x^2 \qquad y(0) = -1 \quad a = 0 \quad b = 1 \quad n = 10$
   c. $y' = 2y - e^x \qquad y(0) = 0 \quad a = 0 \quad b = 1.2 \quad n = 4$
   d. $y' = x - y \qquad y(0) = 1 \quad a = 0 \quad b = 1 \quad n = 4$

1. a. m=2 048 000 b. t=1h

3. v=1.54 $\frac{km}{h}$

4. a. $y = C(2x+1)^{\frac{3}{2}}$ b. $y = \pm\sqrt{2Cx + 3C - 2}$ c. $y = -1 + C\,x(x-2)$

d. $y = C\frac{\sqrt{1-x^2}}{x+1}$

5. a. $y = \frac{3e^x-1-x}{1+x}$ b. $y = x$ c. $y = e^{\frac{4}{3}x-\frac{1}{3}}$

6. a. $y = Ce^{2x} - \frac{5}{2}$ b. $y = Ce^{-3x} + \frac{1}{5}e^{2x}$ c. $y = Ce^{-3x} + \frac{1}{5}e^{2x}$

d. $y = Cx^2 + x^3 - \frac{1}{2}$ e. $y = \frac{C}{x} + \frac{1}{2}x\ln x - \frac{1}{4}x$

7. a. $y = -\frac{5}{2} + \frac{11}{2}e^{2x-2}$ b. $y = \frac{1}{3} + e^{1-x^3}$ c. $y = (x + 1 + \ln 2)e^{-x}$

d. $y = (x^2 + 1)e^{-x^2}$

8. a.

$X5 := \left[0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right]$ $Y5 := [1, 1., 1.080000000, 1.252800000, 1.553472000, 2.050583040]$ $y(x) = -e^{x^2}$

8.b. $X10 := \left[0, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, 1\right]$ $Y10 := [-1, -1., -1.020000000, -1.060800000, -1.124448000, -1.214403840, -1.335844224, -1.496145531, -1.705605905, -1.978502850, -2.334633363]$

$$y(x) = 2 + 2x + x^2 - 3e^x$$

8.c. $X4 := [0, 0.3000000000, 0.6000000000, 0.9000000000, 1.200000000]$ $Y4 := [2, 2., 2.360000000, 3.209600000, 4.942784000]$

$$y(x) = e^x + e^{2x}$$

8.d. $y(x) = -1 + x + 2e^{-x}$ $X4 := \left[0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right]$

$Y4 := [1, 1., 1.125000000, 1.406250000, 1.933593750]$

Engineering Mathematics 3

Differential equations2

Incomplete second order ODE

1. Find the general solution

a. $y'' = x + \sin x$
b. $y'' = \ln x$
c. $xy'' = y'$
d. $y'' = y' + x$
e. $y'' + 2y' = e^x$
f. $y'' = \frac{y'}{x} + x$
g, $2xy'y'' = (y')^2 + 1$
h. $y''(1-y) + 2(y')^2 = 0$
i. $(y')^2 + 2yy'' = 0$
j. $yy'' = 1 + (y')^2$

2. Find the particular solution

a. $y'' = y' + e^x$ $\quad y(0) = -1 \quad y(1) = e$
b. $y''y\ln y = -(y')^2$ $\quad y(0) = e \quad y'(0) = 2$
c. $y''(x^2+1) = 2xy'$ $\quad y(0) = 1 \quad y'(0) = 3$
d. $y'' = \frac{1}{\sqrt{1-x^2}}$ $\quad y(0) = 2 \quad y(1) = \frac{\pi}{2}$
e. $yy'' = 2(y')^2$ $\quad y(0) = 1 \quad y(1) = 2$

Linear, second order ODE with constant coefficients

3. Find the general solution

a. $y'' - y' - 2y = 0$ $\quad y(0) = 0,\ y'(0) = 1$
b. $2y'' + 5y' - 3y = 0$ $\quad y(0) = 1,\ y(2) = e$
c. $y'' - 16y' + 64y = 0$ $\quad y(0) = 0,\ y'(0) = -1$
d. $3y'' - 2y' + y = 0$ $\quad y(0) = 0,\ y'(0) = -1$
e. $9y'' - 6y' + y = 0$ $\quad y(0) = 0,\ y'(0) = -1$
f. $y'' - 4y' + 13y = 0$ $\quad y(0) = 1, y'(0) = -2$
g. $y'' - y' - 6y = 0$ $\quad y(0) = 2,\ y(1) = 1$
h. $y'' - 2y' + y = 0$ $\quad y(0) = \frac{1}{e},\ y(1) = 1 + e$

4. Find the particular solution

a. $y'' + y' - 2y = 2x^2 - 3$ $\quad y_p = Ax^2 + Bx + C$
b. $y'' + 2y' + 2y = 17cos3x$ $\quad y_p = Asin3x + Bcos3x$
c. $y'' - 3y' - 4y = 6e^{2x}$ $\quad y_p = Ae^{2x}$
d. $y'' - y' - 2y = 2e^{3x} - 4x^2 + 8x$ $\quad y_p = Ae^{3x} + Bx^2 + Cx + D$
e. $y'' + 9y = 13xe^{2x}$ $\quad y_p = Ae^{2x}(Bx + C)$

5. Find the general solution (resonance case)

a. $3y'' + 5y' - 2y = 7e^{\frac{1}{3}x} - 4x$ $\quad y_p = Ax + B + Ce^{\frac{1}{3}x}x$
b. $y'' - 3y' = 12x + 8$ $\quad y_p = (Ax + B)x$
c. $9y'' - 6y' + y = 36e^{\frac{1}{3}x}$ $\quad y_p = Ae^{\frac{1}{3}x}x^2$

1.a. $y(x) = \frac{1}{6}x^3 - \sin(x) + \_C1\,x + \_C2$

b. $y(x) = \frac{1}{2}x^2\ln(x) - \frac{3}{4}x^2 + \_C1\,x + \_C2$

c. $y(x) = \_C1 + \_C2\,x^2$

d. $y(x) = -\frac{1}{2}x^2 + e^x\,\_C1 - x + \_C2$

e. $y(x) = \frac{1}{3}e^x - \frac{1}{2}e^{-2x}\,\_C1 + \_C2$

f. $y(x) = \frac{1}{3}x^3 + \frac{1}{2}\,\_C1\,x^2 + \_C2$

g. $y(x) = \frac{2}{3}\,\frac{(-1 + \_C1\,x)^{3/2}}{\_C1} + \_C2,\ \ y(x) = -\frac{2}{3}\,\frac{(-1 + \_C1\,x)^{3/2}}{\_C1} + \_C2$

h. $y(x) = \frac{-1 + \_C1\,x + \_C2}{\_C1\,x + \_C2}$

i. $\frac{2}{3}y(x)^{3/2} - \_C1\,x - \_C2 = 0$

j. $y(x) = \frac{1}{2}\,\_C1\left(\frac{1}{\left(e^{-\frac{x}{\_C1}}\right)^2\left(e^{-\frac{\_C2}{\_C1}}\right)^2} + 1\right)e^{-\frac{x}{\_C1}}\,e^{-\frac{\_C2}{\_C1}},\ y(x) = \frac{1}{2}\,\frac{\_C1\left(\left(e^{-\frac{x}{\_C1}}\right)^2\left(e^{-\frac{\_C2}{\_C1}}\right)^2 + 1\right)}{e^{-\frac{x}{\_C1}}\,e^{-\frac{\_C2}{\_C1}}}$

2. a. $y(x) = \left(-1 + \frac{e}{e-1} + x\right)e^x - \frac{e}{e-1}$

b. $y(x)(\ln(y(x)) - 1) = 2x$

c. $y(x) = 1 + \frac{1}{2}x^3 + \frac{3}{2}x$

d. $y(x) = x\arcsin(x) + \sqrt{1-x^2} - x + 1$

e. $y(x) = -\frac{2}{x-2}$

3.a. $y = -\frac{1}{3}e^{-x} + \frac{1}{3}e^{2x}$ b. $y = e^{\frac{1}{2}x}$ c. $y = e^{8x} - 9xe^{8x}$

d. $y(x) = -\frac{3}{2}\sqrt{2}\,e^{\frac{1}{3}x}\sin\left(\frac{1}{3}\sqrt{2}\,x\right)$ e. $y = e^{\frac{1}{3}x}(1 - \frac{4}{3}x)$

f. $y = -\frac{4}{3}e^{2x}\sin 3x + e^{2x}\cos 3x$ g. $y = \frac{(1-2e^3)e^{-2x} + (2e^{-2}-1)e^{3x}}{e^{-2} - e^3}$

h. $y = e^{x-1} + xe^x$

4. a. $y = C_1e^x + C_2e^{-2x} - x - x^2$ b. $y = C_1e^{-x}\sin x + C_2e^{-x}\cos x + \frac{6}{5}\sin 3x - \frac{7}{5}\cos 3x$

c. $y = C_1e^{4x} + C_2e^{-x} - e^{2x}$ d. $y = C_1e^{2x} + C_2e^{-x} + \frac{1}{2}e^{3x} + 2x^2 - 6x + 5$

5.a. $y = C_1e^{-2x} + C_2e^{\frac{1}{3}x} - \frac{3}{7}e^{\frac{1}{3}x} + 2x + 5 + xe^{\frac{1}{3}x}$ b. . $y = C_1\frac{1}{3}e^{3x} + C_2 - 2x^2 - 4x$

c. $y = C_1e^{\frac{1}{3}x} + C_2xe^{\frac{1}{3}x} + 2x^2e^{\frac{1}{3}x}$

Engineering Mathematics 3.

Differential equations3

Second order ODE Euler method

1. Find the numerical solution on the given interval
   a. $3yy'' + (y')^2 = 0$ $\quad y(0) = 1$ $\quad y'(0) = 3$ $\quad [a,b] = [0,2]$ $\quad n = 5$
   b. $y'' = -y + y'$ $\quad y(0) = 1$ $\quad y'(0) = 0$ $\quad [a,b] = [0,1]$ $\quad n = 4$
   c. $y'' + xy = 0$ $\quad y(1) = 1$ $\quad y'(1) = 3$ $\quad [a,b] = [1,3]$ $\quad n = 6$

Laplace transformation

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$

$f(t)$	$F(s)$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

2. Find the Laplace transform
   a; $f(t) = 1$ $\quad$ b; $f(t) = t^2$ $\quad$ c; $f(t) = e^{-3t}$ $\quad$ d. $f(t) = \sin 2t$
   e; $f(t) = \cos\frac{t}{3}$ $\quad$ f. $H(t) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ t \geq 0 \end{cases}$ (Heaviside or unit step function)
   g; $H_4(t) = f(t) = \begin{cases} 0 & if\ t < 4 \\ 1 & if\ t \geq 4 \end{cases}$ express it using Heaviside function
   h; $H_2(t) - H_4(t) = f(t) = \begin{cases} 0 & if\ t < 2 \\ 1 & if\ 2 \leq t < 4 \\ 0 & if\ t \geq 4 \end{cases}$ (step up step down function) express it using Heaviside function
3. Find the inverse Laplace transform
   a. $F(s) = \frac{s-1}{(s-2)(s+1)}$ $\quad$ b. $F(s) = \frac{2}{(s^2+1)(s^2+4)}$ $\quad$ c. $F(s) = \frac{s-9}{s^2-6s+5}$
   d. $F(s) = \frac{-3s^2-12s-8}{(s+2)^3}$ $\quad$ e. $F(s) = \frac{2s-5}{s^2+4s+8}$ $\quad$ f. $F(s) = \frac{s^2-6s+10}{(s-2)(s-1)(s-3)}$
4. Using the function show that
   $L\{\dot{f}(t)\} = sL\{f(\mathrm{t})\} - f(0)$ and $L\{\ddot{f}(t)\} = s^2L\{f(\mathrm{t})\} - \mathrm{s}f(0) - \dot{f}(0)$
   a; $f(t) = 1$ $\quad$ b; $f(t) = e^{-3t}$ $\quad$ b; $f(t) = 2t$
5. Solve the ODE-s using Laplace transformation
   a. $y'' - 6y' + 5y = 0$ $\quad y(0) = 1,\ y'(0) = -3$
   b. $y' + 2y = 4te^{-2t}$ $\quad y(0) = -3$
   c. $y'' - 3y' + 2y = e^{3t}$ $\quad y(0) = 1,\ y'(0) = 0$
   d. $y'' + 4y = f(t)$ $\quad y(0) = 3,\ y'(0) = -2$ $\quad f(t) = \begin{cases} 1 & if\ 0 \leq t < 4 \\ 0 & if\ t \geq 4\ or\ t < 0 \end{cases}$
   e. $2y'' + y' + 2y = H_5(t) - H_{20}(t)$ $\quad y(0) = 0,\ y'(0) = 0$
   f. $2y'' + y' + 2y = \delta(t-5)$ $\quad y(0) = 0,\ y'(0) = 0$

Solution

1.a. $X := \left[0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, 2\right]$ $Y := [1., 2.200000000, 2.920000000, 3.561454545, 4.155938322, 4.717344700]$

b. $X := \left[0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right]$ $Y := [1., 1., 0.9375000000, 0.7968750000, 0.5625000000]$

c. $X := \left[1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3\right]$ $Y := [1., 2., 2.888888889, 3.481481481, 3.539094650, 2.823045267, 1.189452827]$

2. a. $F(s) = \frac{1}{s}$ b. $F(s) = \frac{2}{s^3}$ c. $F(s) = \frac{1}{3+s}$

d. $F(s) = \frac{2}{s^3+4}$ e. $F(s) = \frac{9s}{9s^3+1}$ g. $F(s) = \frac{e^{-4s}}{s}, f(t) = H(t-4)$

h. $F(s) = \frac{e^{-4s}-e^{-2s}}{s}$, $f(t) = H(t-2) - H(t-4)$

3 a $f := \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$ b. $f := -\frac{1}{3} \sin(2t) + \frac{2}{3} \sin(t)$ c. $f := -e^{5t} + 2e^{t}$

d. $f := (2t^2 - 3) e^{-2t}$ e. $f := \frac{1}{2} e^{-2t} (4\cos(2t) - 9\sin(2t))$ f. $f := \frac{5}{2} e^{t} - 2e^{2t} + \frac{1}{2} e^{3t}$

5.a. $2e^{t} - e^{5t}$ b. $(2t^2 - 3) e^{-2t}$ c. $\frac{5}{2} e^{t} - 2e^{2t} + \frac{1}{2} e^{3t}$

d. $\frac{d^2}{dt^2} y(t) + 4y(t) = \text{Heaviside}(t) - \text{Heaviside}(-4+t)$

$y(t) = -\sin(2t) + 3\cos(2t) + \frac{1}{4}\text{Heaviside}(t) - \frac{1}{4}\text{Heaviside}(-4+t) + \frac{1}{4}\text{Heaviside}(-4+t)\cos(-8+2t) - \frac{1}{4}\cos(2t)\,\text{Heaviside}(t)$

Engineering Mathematics 3

Determinant, adjoint and inverse of matrices, Cramer's rule, Gauss-Jordan elimination

CAS practices

Download *Determinant, gauss elimination.mw* file, solve the exercises.

with(LinearAlgebra): Determinant, ColumnOperation, RowOperation, Adjoint, GenerateMatrix, GaussianElimination, ReducedRowEchelonForm, BackwardSubstitution

Paper work

1. Evaluate the determinants of the matrices:

$$A=\begin{bmatrix}3 & 5\\ -2 & 4\end{bmatrix}\qquad B=\begin{bmatrix}1 & -2 & 3\\ 6 & 7 & -1\\ -3 & 1 & 4\end{bmatrix}\qquad C=\begin{bmatrix}a-3 & 5\\ -3 & a-2\end{bmatrix}\qquad E=\begin{bmatrix}c & -4 & 3\\ 2 & 1 & c^2\\ 4 & c-1 & 2\end{bmatrix}$$

2. Find all values of λ for which $detA=0$

$$A=\begin{bmatrix}\lambda-2 & 1\\ -5 & \lambda+4\end{bmatrix}\qquad A=\begin{bmatrix}\lambda-4 & 4 & 0\\ -1 & \lambda & 0\\ 0 & 0 & \lambda-5\end{bmatrix}$$

3. Decide whether the given matrix is invertible, and if so, use the adjoint method to find its inverse.

$$A=\begin{bmatrix}2 & 5 & 5\\ -1 & -1 & 0\\ 2 & 4 & 3\end{bmatrix}\qquad A=\begin{bmatrix}2 & -3 & 5\\ 0 & 1 & -3\\ 0 & 0 & 2\end{bmatrix}$$

4. Solve the next linear equation systems (LES) by Cramer's rule, where it applies.

$$\begin{aligned}7x_1-2x_2&=3\\ 3x_1+x_2&=5\end{aligned}\qquad \begin{aligned}-4x+5y&=2\\ 11x+y+2z&=3\\ x+5y+2z&=1\end{aligned}\qquad \begin{aligned}x_1-3x_2+x_3&=4\\ 2x_1-x_2&=-2\\ 4x_1-3x_3&=0\end{aligned}$$

5. By examining the determinant of the coefficient matrix, show that the following system has a nontrivial solution if and only if α=β.

$$\begin{aligned}x+y+\alpha z&=0\\ x+y+\beta z&=0\\ \alpha x+\beta y+z&=0\end{aligned}$$

6. Solve by Gauss-Jordan elimination

$$\begin{aligned}x_1+x_2+2x_3&=8\\ -x_1-2x_2+3x_3&=1\\ 3x_1-7x_2+4x_3&=10\end{aligned}\qquad \begin{aligned}x+y+2z&=1\\ 4x+4y+5z&=6\\ 7x+7y+8z&=10\end{aligned}\qquad \begin{aligned}x_1+3x_2-2x_3+2x_5&=0\\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6&=-1\\ 5x_3+10x_4+15x_6&=5\\ 2x_1+6x_2+8x_4+4x_5+18x_6&=6\end{aligned}$$

7. Determine the values of *a* for which the system has no solutions, exactly one solution, or infinitely many solutions.

$$\begin{aligned}x+2y&=1\\ 2x+(a^2-14)y&=a-1\end{aligned}\qquad \begin{aligned}x+2y-3z&=4\\ 3x-y+5z&=2\\ 4x+y+(a^2-14)z&=a+2\end{aligned}$$

Engineering Mathematics 3

Vector space, elementary change of basis

1. Let V be the set of all ordered pairs of real numbers, and consider the following „addition” and „scalar multiplication” operations on **u**=(u_1,u_2) and **v**=(v_1,v_2): **u**+**v**=(u_1+v_1,u_2+v_2), k**u**=(0,ku_2).
   1.1. Compute **u**+**v** and k**u** for **u**=(-1,2), **v**=(3,4) and k=3.
   1.2. In words, explain why V is closed under addition and scalar multiplication.
   1.3. Show that Axiom 10 fails and hence V is not a vector space under the given operations.
2. Show that the set of all 2x2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$with the standard matrix addition and scalar multiplication is a vectorspace.
3. Show that lines through the origin are subspaces of $\mathbb{R}^2$and of $\mathbb{R}^3$.
4. Show that the set of all points (x, y) in for which $x \geq 0$ and $y \geq 0$ is not the subspace of $\mathbb{R}^2$.
5. Show that the set of function with a continuous derivative is a subspace of continous functions.
6. Determine whether the following vectors are linearly independent or linearly dependent in $\mathbb{R}^3$. Find the dimension of S={**a,b,c**}. Construct matrixes where the vectors are in the rows or in the column. Find the rank of the matrixes.
   6.1. **a**=(1,2,3), **b**=(5,6,-1), **c**=(3,2,1) in $\mathbb{R}^3$
   6.2. **a**=(1,2,3), **b**=(5,6,-1), **d**=(-3,-2,7) in $\mathbb{R}^3$
   6.3. **a**=(1,2), **b**=(5,6), **c**=(3,2) in $\mathbb{R}^2$
7. Find the coordinate vector of **w** relative to the basis S={**a,b**}
   7.1. **a**=(1,0), **b**=(0,1), **w**=(3,-7)
   7.2. **a**=(2,-4), **b**=(3,8), **w**=(1,1)
   7.3. **a**=(1,1), **b**=(0,2), **w**=(x,y)
8. Show that **a**=(1,2,1), **b**=(2,9,0), **c**=(3,3,4) could be a basis for $\mathbb{R}^3$.
   8.1. Find the coordinate vector of **v**=(5,-1,9) relative to the basis S={**a,b,c**}.
   8.2. Find the **w** vector in $\mathbb{R}^3$={$\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{e}_3$} standard basis whose coordinate relative to S is $\mathbf{w}_S$=(-1,3,2).
9. Find the coordinate vector of **w** relative to the basis S={**a,b,c**}
   9.1. **a**=(3,3,3), **b**=(1,0,0), **c**=(2,2,0), **w**=(2,-1,3),
   9.2. **a**=(1,2,3), **b**=(-4,5,6), **c**=(7,-8,9), **w**=(5,-12,3)
10. Solve Exercise 9 using elementary change of basis.
11. Let **a**=(-1,1,3), **b**=(-4,-2,0), **c**=(-7,-5,-3) be vectors.
    11.1. Find the dimension of S={**a,b,c**}.
    11.2. If **v**=(-10,-8,-6), **w**=(-10,-8,-6) , is the vektor **v** or **w** in the space generated by S?
12. Find the matrix inverse using elementary basis transformation.

$$A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

Engineering Mathematics 3

Application of change of basis, eigenvalue, eigenvector

1. Solve the next linear equation systems (LES) by change of basis.

$x_1+3x_2+x_3-x_4=7;$ $2x_1+5x_2-x_3+2x_4=22;$ $3x_1+8x_2+x_3-x_4=24.$	$x_1-x_2+x_3-3x_4=-2;$ $x_1-2x_2+3x_3-4x_4=-6;$ $3x_1+4x_2-x_3+2x_4=12;$ $-2x_1+3x_2+2x_3+x_4=2.$	$x_1-x_2+x_3=4;$ $x_1+2x_2+x_3=13;$ $2x_1+4x_2+2x_3=26;$ $4x_1+5x_2+4x_3=43.$
$x+3y+4z=19;$ $x+4y+3z=18$ $x+3y+3z=16$	$x_1+3x_2+x_3-x_4=7;$ $2x_1+5x_2-x_3+2x_4=22;$ $3x_1+8x_2+x_3-x_4=24.$	$x_1-2x_2+3x_3-x_4+2x_5=2;$ $3x_1-x_2+5x_3-3x_4-x_5=6;$ $2x_1+x_2+2x_3-2x_4-3x_5=8.$

2. Determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

$$\begin{aligned} x+\qquad\quad 2y&=1\\ 2x+(a^2-14)y&=a-1 \end{aligned} \qquad \begin{aligned} x+2y-3z&=4\\ 3x-y+5z&=2\\ 4x+y+(a^2-14)z&=a+2 \end{aligned}$$

3. Show that the eigenvalues of the diagonal matrices are the diagonal entries, if
$$B=\begin{bmatrix}3&2&-1\\0&8&-1\\0&0&2\end{bmatrix},\quad C=\begin{bmatrix}3&0&0\\2&8&0\\1&0&2\end{bmatrix}.$$

4. Show that the product of eigenvalues equals to the determinant of the matrix, if
$$A=\begin{bmatrix}2&0&-3\\0&2&3\\-3&3&-2\end{bmatrix}, B=\begin{bmatrix}1&0&2\\0&1&0\\0&0&2\end{bmatrix}$$

5. Show that the eigenvectors of a real, symmetric matrix with distinct eigenvalues are mutually orthogonal if $A=\begin{bmatrix}3&2&2\\2&2&0\\2&0&4\end{bmatrix}$

Engineering Mathematics 3

Applications of Eigenvectors and Eigenvalues.

1. Show that the eigenvectors of symmetrical matrixes are orthogonal to each other.
$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix} \; D = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$
2. Are the next matrixes diagonalizable or not. If the answer is yes, give the diagonal matrix D, and the P, for which $matrix = PDP^{-1}$
$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$
3. Solve the next systems:

$y_1' = y_1 + 4y_2$ $y_2' = 2y_1 + 3y_2$ $y_1(0) = y_2(0) = 0$	$y_1' = 4y_1 + y_3$ $y_2' = -2y_1 + y_2$ $y_3' = -2y_1 + y_3$ $y_1(0) = -1, y_2(0) = 1, y_3(0) = 0$
$y_1' = 2y_1 + 9y_2$ $y_2' = y_1 + 2y_2$ $y_1(0) = 1, y_2(0) = 1$	$y_1' = y_1 + 2y_2 - y_3$ $y_2' = y_1 + y_3$ $y_3' = 4y_1 - 4y_2 + 5y_3$ $y_1(0) = 1, y_2(0) = 1, y_3(0) = 1$

4. Express the quadratic form in matrix notation $\boldsymbol{x}^T A \boldsymbol{x}$, where A is symmetric:

$2x^2 + 6xy - 5y^2;$ $\qquad x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3;$

5. Find the orthogonal change of variable that eliminates the cross product terms in the quadratic form, and express Q in terms of the new variable:

$Q = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3;$ $\qquad Q = 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$

6. Identify the conic by rotating the *xy*-axes to put the conic in standard position. Find the angle Θ through which you rotated the xy-axes.

$5x^2 - 4xy + 8y^2 - 36 = 0$ $\qquad 2x^2 - 4xy - y^2 + 8 = 0$

7. Identify the quadratic rotating the *xyz*-axes to put the qudratic in standard position.

$4x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 4yz = 1$ $\qquad 2x^2 + y^2 + z^2 + 2xy + 2xz = 1$

Engineering Mathematics 3

Function series

1. Find the Fourier series generated by $f(x)$.

$$f(x) = \begin{cases} 1 \ \ if \quad 0 \le x < \pi \\ -1 \quad if \ -\pi \le x < 0 \end{cases}, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$

$$f(x) = \frac{x}{2} \ \ if \ -\pi \le x < \pi, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$

$$f(x) = \begin{cases} \frac{\pi}{2} + x \ if \quad 0 \le x < \pi \\ \frac{\pi}{2} - x \ if \ -\pi \le x < 0 \end{cases}, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$

$$f(x) = |x| \ \ if \ -\pi \le x < \pi, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$

$$f(x) = \begin{cases} x - 1 \ \ if \quad 0 \le x < \pi \\ -1 \quad if \ -\pi \le x < 0 \end{cases}, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$

$$f(x) = x^2 \ \ if \ -\pi \le x < \pi, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$

$$f(x) = \begin{cases} -\frac{\pi}{2} \quad if \quad -\pi \le x < -\frac{\pi}{2} \\ x \quad if \quad -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ \frac{\pi}{2} \quad if \quad \frac{\pi}{2} \le x < \pi \end{cases}, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$

$$f(x) = \begin{cases} -2x + 1 \ \ if \quad -\pi \le x < 0 \\ 2x + 1 \quad if \quad 0 \le x < \pi \end{cases}, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$

$$f(x) = \begin{cases} \frac{x}{2} + 1 \ \ if \quad -\pi \le x < 0 \\ 1 \quad if \quad 0 \le x < \pi \end{cases}, \qquad f(x + k2\pi) = f(x), \qquad k \in \mathbb{Z}$$